The London Equations

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Abstract: The London Equations were the first successful attempt at characterizing the electrodynamic behavior of superconductors. In this paper we review the motivation, derivation, and modification of the London equations since 1935. We also mention further developments by Pippard and others. The shaping influences of experiments on theory are investigated.

Introduction:

Following the discovery of superconductivity in supercooled mercury by Heike Onnes in 1911, and its subsequent discovery in various other metals, the early pioneers of condensed matter physics were faced with a mystery which continues to be unraveled today. Early experiments showed that below a critical temperature, a conventional conductor could abruptly transition into a superconducting state, wherein currents seemed to flow without resistance. This motivated a simple model of a superconductor as one in which charges were influenced only by the Lorentz force from an external field, and not by any dissipative interactions. This would lead to the following equation:

$$\frac{m}{ne^2}\frac{dJ}{dt} = E \tag{1}$$

Where m, e, and n are the mass, charge, and density of the charge carriers, commonly taken to be electrons at the time this equation was being considered. In the following we combine these variables into a new constant both for convenience and because the mass, charge, and density of the charge carriers are often subtle concepts.

$$\Lambda = \frac{m}{ne^2}$$

This equation, however, seems to imply that the crystal lattice which supports superconductivity is somehow invisible to the superconducting charges themselves. Further, to suggest that the charges flow without friction implies, as the Londons put it, "a premature theory." Instead, the Londons desired a theory more in line with what Gorter and Casimir had conceived [3], whereby a persistent current is spontaneously generated to minimize the free energy of the system. This would not be unlike a ferromagnetic transition, which spontaneously sources a persistent magnetic field when cooled below a certain temperature.

Heinz and Fritz London were motivated to develop their theory of the superconductor by a crucial experiment by Meissner and Ochsenfeld. In 1933 Meissner and Ochsenfeld discovered that in the superconducting state, a metal would expel any magnetic field from its interior. In this sense, a superconductor could be thought of as a perfect diamagnet, and indeed is perhaps better thought of as a perfect diamagnet than as a perfect conductor, as this would imply that any magnetic field present in the metal past the superconducting transition would be frozen inside rather than being violently expelled. It is this phenomenon, now known as the Meissner effect, which the Londons sought to explain in their paper "The Electromagnetic Equations of the Supraconductor."

The London Equations:

The Londons began by considering the implications of Equation 1, which they called “the acceleration equation.” If this were true, then Faraday’s Law implies

$$\nabla \times \left(\Lambda \frac{dJ}{dt}\right) = -\frac{dB}{dt} \tag{2}$$

Neglecting the displacement current, this yields

$$\nabla^2 (B - B_0)\frac{\Lambda}{\mu_0} = B - B_0 \tag{3}$$

This second order vector equation is almost familiar. Crucially, one may note that its solutions which behave regularly inside the superconductor decay exponentially to B_0 as follows:

$$\frac{B(x) - B_0}{B_{ext} - B_0} = e^{-\frac{x}{\lambda_L}}$$
$$where \quad \lambda_L = \sqrt{\frac{m}{\mu_0 n e^2}} \tag{4}$$

The Londons estimated lambda to be around 10^-7 m – less than a micron! This would mean that the magnetic field inside a superconductor quickly decays to B_0 essentially regardless of the external field strength. But from Meissner and Ochsenfeld’s experiment, we must conclude that $B_0 = 0$. The parameter λ_L, now known as the London penetration depth, therefore quantifies how far magnetic fields can seep into a superconductor, and since the decay is exponential, it implies a steep cutoff. The Londons therefore proposed the following as an equation analogous to Ohm’s law for superconductors.

$$\Lambda \nabla \times J = -B \tag{5}$$

But this equation alone cannot explain the response of a current to an electric field. Reversing the above derivation, we find instead of Equation 1 that we must have

$$\Lambda \frac{dJ}{dt} + \nabla f = E \tag{6}$$

But if we view this as an antisymmetric tensor equation, we are forced to identify the new object f as the time component of the current. But the time component of the current is the charge, and its gradient is the divergence of E!

$$\Lambda \left(\frac{dJ}{dt} + \nabla \rho\right) = E \tag{7}$$

These Equations 5 and 7 are almost the London equations. We will see soon that a slight modification must first be made. Ignoring that, with a little algebra Equation 7 can be rewritten as

$$\Lambda\, \nabla^2 E = E \tag{8}$$

This is the same equation that the magnetic field satisfies. In fact it is easy to show that the ten electromagnetic quantities E, B, ρ, and J all satisfy this equation, thus they all decay to zero inside of the superconductor with the London penetration depth. That is to say, current only flows on the surface of a superconductor. The Londons also mention that if we do not neglect the displacement current, the following equations hold

$$\Lambda\left(\nabla^2 E - \frac{d^2 E}{dt^2}\right) = E \tag{9a}$$

$$\Lambda\left(\nabla^2 B - \frac{d^2 B}{dt^2}\right) = B \tag{9b}$$

$$\Lambda\left(\nabla^2 J - \frac{d^2 J}{dt^2}\right) = J \tag{9c}$$

$$\Lambda\left(\nabla^2 \rho - \frac{d^2 \rho}{dt^2}\right) = \rho \tag{9d}$$

A Later Development:

It is worth noting that the leap from Equation 6 to 7 is by no means trivial. In their original paper, the Londons showed that by identifying the mysterious gradient term with the charge density their equations could be put into a nice antisymmetric tensor. In other words, they appealed to every physicist's desire for things to be maximally beautiful. However, another reasonable possibility is that the gradient term is zero.

In 1935, several months after the publication of "The Electromagnetic Equations of the Supraconductor," H. London published a follow up paper based on an experiment he performed on a condenser with superconducting plates [5]. Though we will not go into the details of the experimental setup, H. London found that the gradient term in Equation 6 could not be identified with the charge density as postulated earlier, and in fact was zero up to experimental precision. These results were shortly thereafter endorsed by F. London as well [6], and the equations we call the London equations today are as follows

$$E = \Lambda \frac{dJ}{dt} \tag{10a}$$

$$B = -\Lambda\, \nabla \times \mathrm{J} \tag{10b}$$

Which can also be written in the more compact form

$$A = -\Lambda\, J \tag{10c}$$

The Pippard Equations:

Another significant correction to the London theory came in 1953 from Pippard [9]. In the process of studying the effects on the superconducting state of doping tin with indium impurities, Pippard noticed that though thermodynamic properties such as the critical temperature changed very little, the penetration depth changed drastically. Due to the complicated nature of Pippard's experiment, we will not go into the specifics of how the penetration depth was measured, but with 3% indium doping, Pippard found the penetration depth to double, while the critical temperature shifted only from 3.72 to 3.62 K.

Pippard argued that the introduction of only 3% indium could not possibly have so drastically changed the mass or density of the superconducting electrons, and if it had then the thermodynamic properties of the superconductor would also have had to change accordingly. Therefore, the phenomenological parameter Λ which determines the penetration depth in the London equations could not be identified with m/ne^2 . Instead, inspired by the work of Reuter and Sondheimer in explaining the anomalous skin effect [8], Pippard postulated that the superconducting charges should be thought of as delocalized within a region determined by the mean free path, and as a result the current in a superconductor should be calculated from an appropriate average of the field over this region.

In Reuter and Sondheimer's explanation of the anomalous skin effect, they note that in the case of a spatially varying electric field in a metal, the common expression of Ohm's law $J = \sigma E$ cannot be valid. Instead, because the electron bumbles about in a volume of characteristic length given by the mean free path, the current density at any point within the metal must instead be taken as an appropriate average of the electric field smeared over the volume within which the electron is localized. Reuter and Sondheimer therefore found that the following equation must be used in the case where charge delocalization was significant:

$$J = \frac{3\sigma}{4\pi l} \int \frac{r(r \cdot E)e^{-r/l}}{r^4} \tag{11}$$

Where l is the mean free path.

Pippard recognized that since doping with impurities would significantly alter the mean free path of the superconducting electrons, the penetration depth anomaly could be a result of the same phenomenon. He therefore postulated that the London equation 10c should be modified to

$$J = -\frac{\xi}{\xi_0 \Lambda} A \tag{12a}$$

$$J = -\frac{3}{4\pi \xi_0 \Lambda} \int \frac{r(r \cdot A)e^{-r/\xi}}{r^4} \tag{12b}$$

Though this equation is not obviously soluble, Pippard notes that it can be solved analytically in the case of a semi-infinite chunk of superconductor with a flat surface. That is to say that $z > 0$ is vacuum and $z < 0$ is superconducting, with A oriented along the x axis. In this case the equation reduces to

$$\frac{d^2A_x}{dz^2} = \frac{3\pi}{\xi_0\Lambda}\int_0^\infty k\left(\frac{z-t}{\xi}\right)A_x(t)dt \tag{13}$$

$$where \quad k(u) = \int_1^\infty \left(\frac{1}{s} - \frac{1}{s^3}\right)e^{-s|u|}ds$$

This differential equation can apparently be solved for the full magnetic field distribution within the superconductor as Reuter and Sondheimer showed, though we do not attempt to do so here. The resulting penetration depth is found to be

$$\lambda = \pi\xi / \int_0^\infty \log\left(1 + \frac{3\pi\xi^3}{\xi_0\Lambda}\frac{\kappa(t)}{t^2}\right) \tag{14}$$

$$where \quad \kappa(t) = \frac{2}{t^3}[(1+t^2)\tan^{-1}t - t]$$

What is the meaning of all this? The key point here is that by including non-point charge effects, we find that the effective penetration depth changes! We can go further and examine this behavior in two limiting cases:

$$\lambda \approx \sqrt{\frac{\xi_0\Lambda}{4\pi\xi}} \qquad when\ \xi \ll \lambda \tag{15a}$$

$$\lambda \approx \left(\frac{\sqrt{3}\xi_0\Lambda}{2\pi^2}\right)^{\frac{1}{3}} \ when\ \xi \gg \lambda \tag{15b}$$

Though we have already said far too much about Pippard's equations, we would finally like to point out that in the wake of BCS theory an equivalent analysis can be applied to the nonlocality induced by the formation of Cooper pairs. Though Pippard's formulation was based on the mean free path change in dirty superconductors, since the charge carried by a Cooper pair cannot be localized to a single point it too contributes to an analogous change in the dispersion length [12].

Other deviations from the London and Pippard phenomenological theories can occur as well. One well known example of this occurs in the case of ac signals, where the characteristic length is known as the Campbell penetration depth [11]. Another possibility suggested by Silaev, Winyard, and Babaev is that multiple London penetration scales can emerge [13]. This was supposedly demonstrated in a multiband London model, though I found their paper frankly *impenetrable*.

Conclusions:

Have the London equations predicted experimental results? Yes. Though the London equations were motivated strongly by the Meissner Ochsenfeld experiments, they did more than postdict results, they also came with a prediction for the field penetration depth in a superconductor. The importance of the penetration depth was recognized at least as early as 1939 experimentally by Appelyard and Bristow, and it continues to be measured in superconductors today [7, 10].

Have the London equations stood the test of time? Apparently yes, with the exception of some modifications. Although our understanding of the origins of superconductivity has come a long way in the past century, these classical electrodynamic equations for the behavior of a superconductor have remained essentially unchanged since 1935. Their utility both experimentally and theoretically is as great today as it was in 1935, and it is likely to persist well into the future.

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